SAMPLE 2021

الشهادة المتوسطة

عدد المسائل: ستة | مسابقة في مادة الرياضيات

المدة ساعتان | إعداد: أ. مازن الخطيب

ارشادات عامة: - يسمح باستعمال آلة حاسبة غير قابلة للبرمجة او اختزان المعلومات او رسم البيانات

- يستطيع المرشح الإجابة بالترتيب الذي يناسبه دون االلتزام بترتيب المسائل الوارد في المسابقة.

I. (2 points)

In the table below, only one among the proposed answers to each question is correct. Write down the number of each question and give, with justification, the corresponding answer for your choice.

	QUESTION	PROPOSED ANSWERS		
		a	b	c
1	$\left(\sqrt{3}-1\right)^2\left(\sqrt{3}+1\right)^2$ is equal to:	2	4	8
2	The solution of the following system $\begin{cases} x+2y=3 \\ y=-x+3 \end{cases}$ is :	$\{(x,y)=(3,0)\}$	$\{(x,y)=(0,3)\}$	$\{(x,y)=(-2,1)\}$
3	Given: $ab=1-\sqrt{3}$ and $a+b=1+\sqrt{3}$. Then: $\frac{1}{a}+\frac{1}{b}=$	$-\sqrt{3}-2$	$1+\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}-1}{2}$
4	Given $x+\frac{2}{x}=4$. Then, $x^2+\frac{4}{x^2}=$	16	12	4

II. (2.5 points)

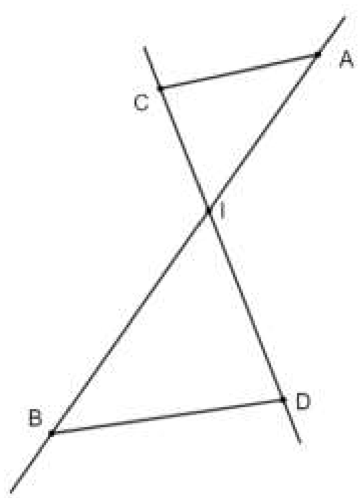
Given: $a=\left(1-\frac{1}{3}\right)^2 \div 3^{-2}$ $\quad b=\left(\sqrt{48}-2\right)-\frac{\sqrt{3}-2}{\sqrt{3}+2}$ $\quad c=\frac{0.021\times\left(0.01\right)^{-2}+0.007\times10^3}{31\times7\times3^{-1}}$

1) Verify that a is a natural number.
2) Show that $b=5$ and calculate c.
3) Given triangle ABC such that $BC=a$, $AC=b$, and $AB=c$.
   - a) What is the nature of triangle ABC.
   - b) Calculate the area of triangle ABC.
   - c) Let H be the foot of the perpendicular issued from B to [AC]. Deduce the length BH.

III. (4 points)

Consider the expressions: $P(x)=4(3x-4)^2-12x+16$ and $Q(x)=25(x-1)^2-\left(x-3\right)^2$

1) Show that $P(x)=4(3x-4)(3x-5)$.
2) Factorize $Q(x)$.
3) Solve $P(x)=Q(x)$ and $Q(x)-16=0$.
4) Let $H(x)=\frac{P(x)}{Q(x)}$.
   - a) For what values of x is $H(x)$ defined?
   - b) Simplify $H(x)$ then solve $H(x)=\frac{5}{4}$.
5) In the adjacent figure, the two straight lines (AB) and (CD) intersecting at I.
   $IA=5$ cm, $AB=3x$, $BD=2x-1$, and $AC=4$ cm.
   - a) Find x when I is the midpoint of [AB].
   - b) Show that $H(x)=\frac{IB}{BD}$. Find x when $\left(AC\right)$ is parallel to $\left(BD\right)$.



IV. (1.5 points)

In grade 9 class, there are 20 students.
Yesterday, 4 boys and 1 girl were absent; the number of boys was then double the number of girls.

1) Show that the previous information can be modeled by the following system: $\begin{cases} x+y=20 \\ x-2y=2 \end{cases}$.

2) Solve the system, and find the number of boys and girls in Grade 9 class.

V. (5 points)

In an orthonormal system of axes (x'ox, y'oy), consider the points $A(-1,3)$, $B(5,1)$, and $C(4,-2)$.
Let (d) be a straight line of equation $y = x+4$.

1) a. Plot A, B and C.
b. Verify that A is a point on the line (d). Draw (d).

2) a. Calculate $AB, AC, and\ BC$.
b. Show that triangle ABC is a right angled triangle.

3) Let (C) be a circle circumscribed about triangle ABC. Determine the coordinates of I, the center of (C) and calculate its radius.

4) Given the two points E and F. E is a point diametrically opposite to B and $F(1,-3)$.
a. Find the coordinates of E.
b. Show that F is a point of circle (C).

5) a. Find the equation of line (AC).
b. Show that (d) is tangent to (C) at A.
c. What is the nature of quadrilateral AEFC? Justify.

VI. (5 points)

In the adjacent figure:

- (C) is a semi-circle with center O, diameter [AB] such that $AB = 6$ cm.
- M is a point on (C) such that $AM = 3\sqrt{3}$ cm.
- (d) is the tangent at B to (C)

1) Copy the figure that will be completed in the remaining parts of the problem.

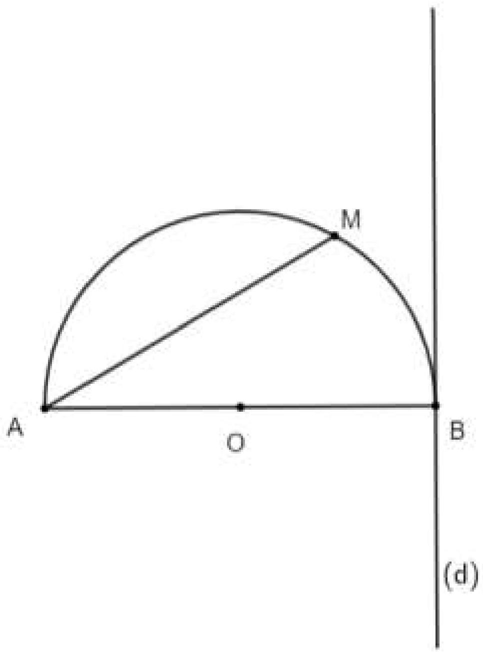
2) a. Calculate BM.
b. Verify that triangle AMB is semi-equilateral triangle and deduce the nature of triangle OMB.

3) The line (OM) cuts (d) at E. The tangent drawn from E to (C) cuts circle (C) at N and (AB) at P.

a. Show that (MN) is parallel to (AB) and (BM) parallel to (ON).
b. What is the nature of quadrilateral ONMB? Justify.

4) a. Show that B,E,N, and O belongs to the same circle (C').
b. Verify that M is the center of (C') and calculate its radius.

5) The line (ON) cuts the line (d) at F.
Show that (OE) is perpendicular to (FP).



Sample 2021 - Correction

Ex.1

1/ $(\sqrt{3}-1)^2(\sqrt{3}+1)^2 = \left[(\sqrt{3}-1)(\sqrt{3}+1)\right]^2$

$= (\sqrt{3}^2-1)^2 = (3-1)^2 = 2^2 = 4$ (b)

2/ $\begin{cases} x+2y=3 \longrightarrow eq\ 1 \\ y=-x+3 \longrightarrow eq\ 2 \end{cases}$

Sub eq 2 in (1): $x+2(-x+3)=3$

$x-2x+6=3$

$-x = 3-6$

$-x=-3$ then $\boxed{x=3}$

Sub x to find y: $y=-3+3=0$

$S=\{(x,y)=(3,0)\}$ (a)

3/ $ab = 1-\sqrt{3}$

$a+b = 1+\sqrt{3}$

$\frac{1}{a}+\frac{1}{b} = \frac{b+a}{ab} = \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \frac{1+2\sqrt{3}+3}{1-\sqrt{3}^2} = \frac{4+2\sqrt{3}}{-2}$

$= \frac{2(2+\sqrt{3})}{-2} = \frac{2+\sqrt{3}}{-1} = -2-\sqrt{3}$ (a)

4/ $x+\frac{2}{x}=4$

$\left(x+\frac{2}{x}\right)^2 = 16$

$x^2+2(x)\frac{2}{x}+\frac{4}{x^2}=16$

$x^2+4+\frac{4}{x^2}=16$

$x^2+\frac{4}{x^2} = 16-4 = 12$ (b).

II.

1/ $a = \left(1-\frac{1}{3}\right)^2 \div 3^{-2} = \left(\frac{3-1}{3}\right)^2 \div \frac{1}{3^2}$

$= \left(\frac{2}{3}\right)^2 \times \frac{9}{1} = \frac{4}{9}\times 9 = 4.$ (natural n°)

2/ $b = \sqrt{48}-2-\frac{\sqrt{3}-2}{\sqrt{3}+2}$.

$\frac{\sqrt{3}-2}{\sqrt{3}+2} = \frac{\sqrt{3}-2}{\sqrt{3}+2}\times\frac{\sqrt{3}-2}{\sqrt{3}-2} = \frac{(\sqrt{3}-2)^2}{\sqrt{3}^2-2^2}$

$= \frac{3-4\sqrt{3}+4}{-1} = \frac{7-4\sqrt{3}}{-1} = -7+4\sqrt{3}.$

$b = \sqrt{48}-2-(-7+4\sqrt{3})$

$= 4\sqrt{3}-2+7-4\sqrt{3} = 5$ verified.

$c = \frac{0.021\times(0.01)^{-2}+0.007\times 10^3}{31\times 7\times 3^{-1}}$

$= \frac{21\times 10^{-3}\times(10^{-2})^{-2}+7\times 10^{-3}\times 10^3}{31\times 7\times 3^{-1}}$

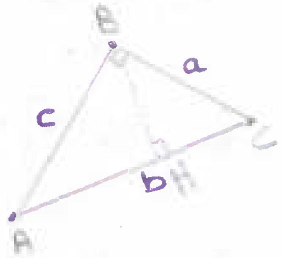
$= \frac{21\times 10^{-3}\times 10^4+7}{31\times 7\times 3^{-1}} = \frac{21\times 10+7}{31\times 7\times 3^{-1}}$

$= \frac{7(3\times 10+1)}{7\times 31\times 3^{-1}} = \frac{\cancel{7}\times\cancel{31}}{\cancel{7}\times\cancel{31}\times 3^{-1}} = 3.$

3/ $AB^2 = c^2 = 3^2 = 9$.

$AC^2 = b^2 = 5^2 = 25$

$BC^2 = a^2 = 4^2 = 16$



$9+16=25$

$AB^2+BC^2 = AC^2$

By converse of pythagorean theorem.

ΔABC is right of vertex B.

b) Area $= \frac{\text{base}\times\text{height}}{2} = \frac{AB\times BC}{2} = \frac{3\times 4}{2}$

$= 6\ u^2.$

c) $\text{Area}_{\Delta ABC} = 6 = \frac{AC\times BH}{2}$

Then $BH = \frac{2\times 6}{AC} = \frac{12}{5} = 2.4$ cm.

Exercise 3

1/ $P(x) = 4(3x-4)^2 - 12x + 16$
$= 4(3x-4)^2 - 4(3x-4)$
$= 4(3x-4)(3x-4-1)$
$= 4(3x-4)(3x-5)$.

2/ $Q(x) = 25(x-1)^2 - (x-3)^2$
$= [5(x-1)-(x-3)][5(x-1)+(x-3)]$
$= (5x-5-x+3)(5x-5+x-3)$
$= (4x-2)(6x-8)$
$= 4(2x-1)(3x-4)$.

3/ * $P(x) = Q(x)$
$P(x) - Q(x) = 0$
$4(3x-4)(3x-5) - 4(2x-1)(3x-4) = 0$
$4(3x-4)(3x-5-2x+1) = 0$
$4(3x-4)(x-4) = 0$
$3x-4 = 0$ or $x-4=0$
$x = 4/3$ or $x = 4$
$S = \{4/3, 4\}$.

* $Q(x) - 16 = 0$
$4(2x-1)(3x-4) - 16 = 0$
$4(6x^2 - 8x - 3x + 4) - 16 = 0$
$24x^2 - 44x + 16 - 16 = 0$
$24x^2 - 44x = 0$
$4x(6x-11) = 0$
$4x = 0$ or $6x - 11 = 0$
$x = 0$ or $x = 11/6$ $S = \{0, 11/6\}$.

4/ $H(x) = \dfrac{P(x)}{Q(x)}$
$= \dfrac{4(3x-4)(3x-5)}{4(2x-1)(3x-4)}$

a) $Q(x) \neq 0$
$2x-1 \neq 0$ and $3x-4 \neq 0$
$x \neq 1/2$ and $x \neq 4/3$
H(x) is defined for all values of x except 1/2 & 4/3.

b) $H(x) = \dfrac{3x-5}{2x-1}$

$H(x) = 5/4$
$\dfrac{3x-5}{2x-1} = \dfrac{5}{4}$
$12x - 20 = 10x - 5$
$12x - 10x = -5 + 20$
$2x = 15$
$x = \dfrac{15}{2}$ accepted.

5/ a) I * [AB].
$AI = \dfrac{AB}{2}$
$5 = \dfrac{3x}{2}$
$x = \dfrac{10}{3}$ cm.

b/ $IB = AB - AI = 3x - 5$.
$\dfrac{IB}{BD} = \dfrac{3x-5}{2x-1} = H(x)$ verified.

(AC) // (BD).
Acc. to Thale's:
$\dfrac{IA}{IB} = \dfrac{IC}{ID} = \dfrac{CA}{BD}$

$\dfrac{5}{3x-5} = \dfrac{IC}{ID} = \dfrac{4}{2x-1}$

then, $\dfrac{3x-5}{2x-1} = \dfrac{5}{4}$
$H(x) = 5/4$.
$x = \dfrac{15}{2}$ cm (part 4-b).

IV.

1/ Let x be the n° of boys
y be the n° of girls

There are 20 students $\longrightarrow$ boy & girls = 20
then $x + y = 20 \longrightarrow$ eq 1

Yesterday, 4 boys were absent $\longrightarrow$
n° of boys $= x - 4$.
1 girl was absent $\longrightarrow$
n° of girls $= y - 1$.
n° of boys was double n° of girls $\longrightarrow (x-4) = 2(y-1)$
$x - 4 = 2y - 2$
$x - 2y = 2 \longrightarrow$ eq 2

So the above info is modeled by the system: $\begin{cases} x+y=20 \\ x-2y=2. \end{cases}$

2/ $x - \begin{cases} x+y=20 \\ x-2y=2 \end{cases}$ $\quad \begin{cases} x+y=20 \\ -x+2y=-2 \end{cases}$

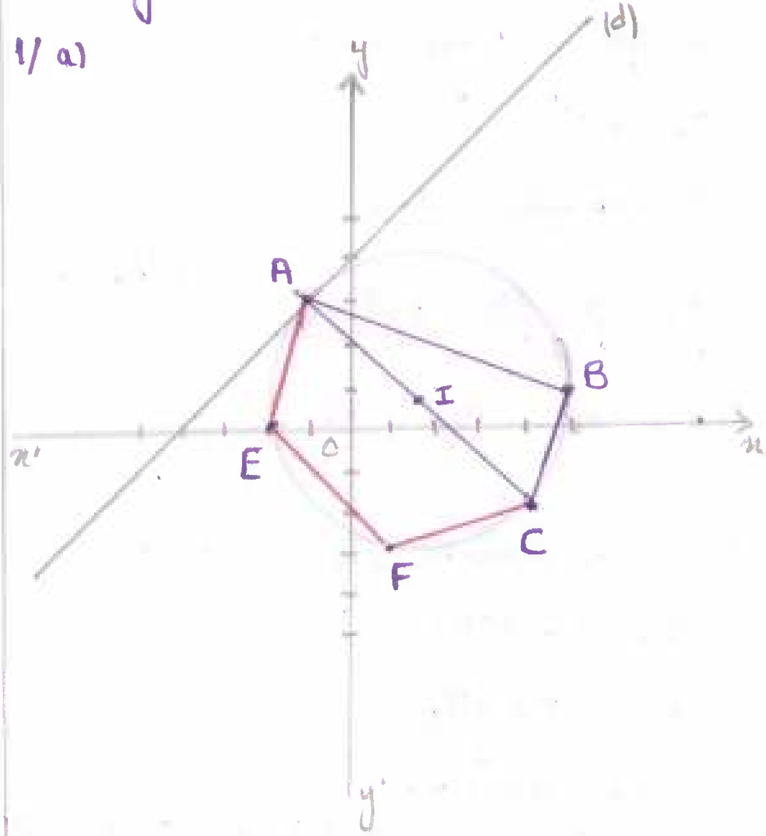
$3y = 18$
$y = 18/3 = 6$.

Sub y to find x:
$x + 6 = 20$
$\boxed{x = 14}$

So n° of boys $= x = 14$.
n° of girls $= y = 6$.

V. $A(-1,3)$ $B(5,1)$ & $C(4,-2)$.
(d): $y = x + 4$

1/ a)



b. Sub A in eq. of (d)
$y_A = x_A + 4$
$3 = -1 + 4$
$3 = 3$ True so $A \in (d)$.

(d): $y = x + 4$ $\quad \begin{array}{c|c} x & y \\ \hline 0 & 4 \end{array}$

2/ $AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$
$= \sqrt{(5+1)^2 + (1-3)^2}$
$= \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$ u.

$AC = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2}$
$= \sqrt{(4+1)^2 + (-2-3)^2} = \sqrt{5^2 + 5^2}$
$= \sqrt{50}$ u. $= 5\sqrt{2}$ u.

$BC = \sqrt{(4-5)^2 + (-2-1)^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$ u.

b. $AB^2 = 40 \quad AC^2 = 50 \quad BC^2 = 10$
$40 + 10 = 50$
$AB^2 + BC^2 = AC^2$
By conv. of pythagoras, $\triangle ABC$ is right at B.

3/ ΔABC is right at B
then center of circumscribed circle is midpt. of hyp [AC].
$I * [AC]$.

$x_I = \frac{x_A + x_C}{2} = \frac{-1+4}{2} = \frac{3}{2}$.

$y_I = \frac{y_A + y_C}{2} = \frac{3+2}{2} = 1/2$. $I(3/2, 1/2)$.

$r = \frac{hyp}{2} = \frac{AC}{2} = \frac{5\sqrt{2}}{2}$ u.

4/ E is diametrically opp. to B
then $I * [EB]$.

$x_I = \frac{x_E + x_B}{2}$

$2x_I = x_E + x_B$

$x_E = 2x_I - x_B = 2\left(\frac{3}{2}\right) - 5 = 3 - 5 = -2$.

$y_I = \frac{y_E + y_B}{2}$

$y_E = 2y_I - y_B = 2\left(\frac{1}{2}\right) - 1 = 0$.

so $E(-2, 0)$.

b) $IF = \sqrt{(x_F - x_I)^2 + (y_F - y_I)^2}$

$= \sqrt{(1 - 3/2)^2 + (-3 - 1/2)^2} = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-7}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{49}{4}} = \sqrt{\frac{50}{4}} = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2}$ = radius

so $F \in (C)$

5/ $(AC): y = ax + b$

$a = \frac{y_C - y_A}{x_C - x_A} = \frac{-2-3}{4+1} = \frac{-5}{5} = -1$.

$(AC): y = -x + b$

Sub A: $3 = -(-1) + b$

$3 = 1 + b$

$\boxed{b = 2}$

so $(AC): y = -x + 2$

b/ $A \in (d)$ & $A \in (C)$
[AC] diameter of (C).

slope (d) × slope (AC) = $1 \times -1 = -1$

then (d) ⊥ (AC)

so (d) is tangent to (C) at A.
(Tangent ⊥ diameter).

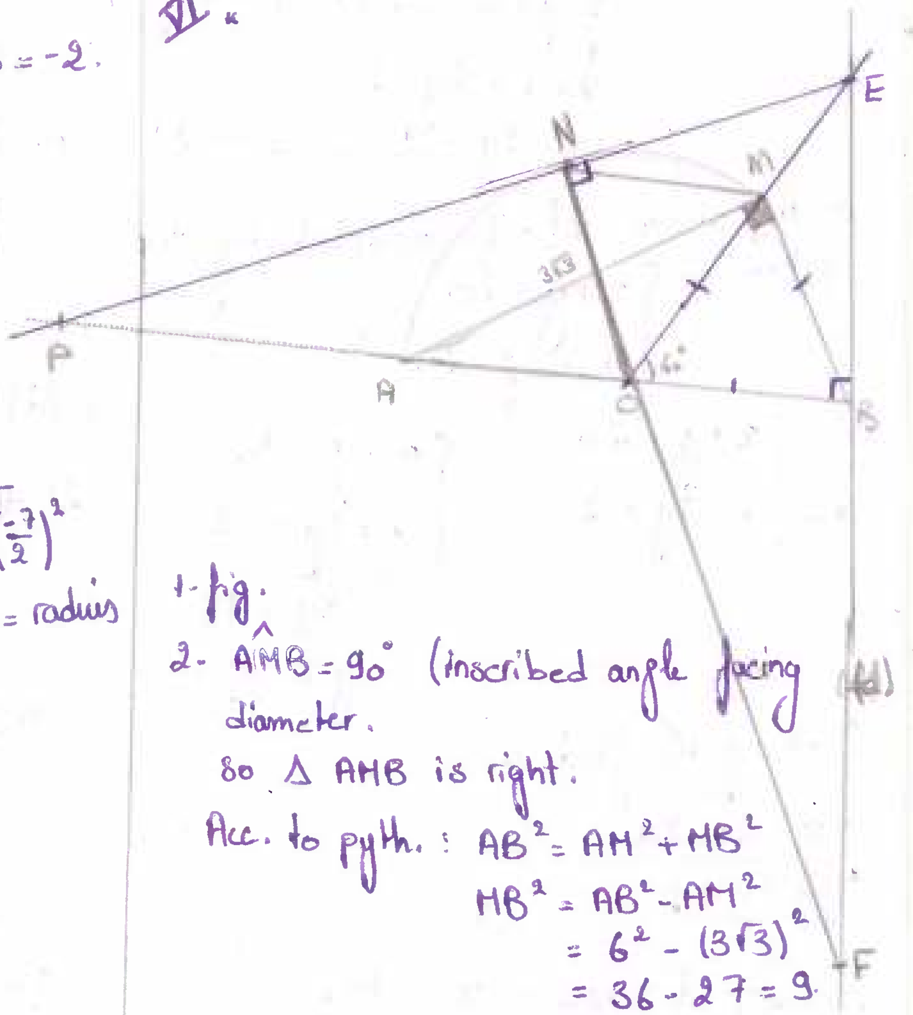
c) $\text{slope}(EF) = \frac{y_F - y_E}{x_F - x_E} = \frac{-3-0}{1+2} = -1$

slope (EF) = slope (AC) = −1

so (EF) // (AC)

so AEFC is a trapezoid.

VI.



1. fig.

2. $\hat{AMB} = 90°$ (inscribed angle facing diameter.
so ΔAMB is right.
Acc. to pyth.: $AB^2 = AM^2 + MB^2$
$MB^2 = AB^2 - AM^2$
$= 6^2 - (3\sqrt{3})^2$
$= 36 - 27 = 9$.

so $MB = \sqrt{9} = 3$ cm.

b) ΔAMB is right at M. (proved)
$MB = 3 = \frac{AB}{2} = \frac{hyp}{2}$.
so [MB] is a side facing (30°)
& thus ΔAMB is semi-equil. triangle.

OM = OB = 3cm (radii of same circle)
MB = 3cm (proved)
OM = OB = MB
so Δ OMB is an equilatural Δ.
(3 equal sides).

3/ a. (EN) & (EB) are two tagents issued from an external point to the same circle (C).
so [ON) is the bisector of $\widehat{NOB}$ (property).
then, $\widehat{NOM} = \widehat{MOB} = 60°$.

ON = OM (radii of same circle)
then, Δ ONM is iso Δ of vertex O.
so $\widehat{ONM} = \widehat{OMN} = \frac{180-60}{2} = 60°$
$\widehat{NMO} = \widehat{MOB} = 60°$
then (MN) // (OB) (alternate interior angles are equal).

$\widehat{AON} = 180° - \widehat{NOM} - \widehat{MOB}$
$= 180° - 60° - 60° = 60°$
$\widehat{ABM} = 60°$ (Δ ABM is semi-eq. Δ)
then, $\widehat{AON} = \widehat{ABM} = 60°$
so (ON) // (BM) (corresponding equal angles)

b. (ON) // (BM)
(MN) // (OB)
so OBMN is a parm (opp. sides are //)
but OB = BM = 3 (proved)
then OBMN is a rhombus.
(parm with adjacent equal sides)

4/a. $\widehat{ONE} = 90°$ (Tangent ⊥ radius)
so N belong to a circle of ~~diam~~ diameter [OE]
$\widehat{OBE} = 90°$ (Tngt ⊥ radius)
so B belong to a circle of diameter [OE].
Then, O, B, N, & E belong to the same circle (C') of diameter [OE].

b. NMBO is a rhombus (proved)
so MN = OB = 3cm.
but OM = ON = OB (radii of same circle)
so MN = OM.

Δ OME is right at N.
NM = MO (proved)
then [NM] is median
(median relative to hyp).
so M * [OE].
& thus M is center of (C').
radius = MO = 3 cm.

5/ In Δ PEF,
(PB) ⊥ (EF) (Tngt ⊥ radius)
so [BP] is the 1st height.
(FN) ⊥ (PE) (Tngt ⊥ radius)
so [FN] is 2nd height.
(PB) ∩ (FN) = {O}, then O is the orthocenter of Δ PEF.
so [OE) is the 3rd height.
& hence (OE) ⊥ (FP).